Chapter 1
Introduction

Dimension, *dimensional quantities*, *dimensionless quantities*, *fundamental quantities*, and *derived quantities* are defined. *Essentials of dimensional analysis* are shown using a problem related to a *simple pendulum*.

1.1 Preliminary Approaches

A series of physical quantities can describe natural phenomena and engineering problems such that the physical laws governing those phenomena and problems can be understood. Revealing those physical laws involves three steps:

Step 1. Classifying *physical quantities* of a given phenomenon or problem according to the *natures* of these physical quantities

Step 2. Finding correlations that connect the physical quantities

Step 3. Finding *causality* that connects the physical quantities

To determine *causality*, it is necessary to understand physical links and relations in a phenomenon or problem. Fundamental principles of physics may then be used to find *parameters of cause and effect* governing the phenomenon or problem. Parameters must be ranked according to importance, and only parameters in the same class can be compared in terms of *magnitude*. Deeper analysis means better results, so the analyst needs rich experience and resourcefulness in order to succeed. Trial and error is the usual way to achieve satisfactory results.

1.2 Dimension

In this book, *dimension* is used to express the essential nature of a quantity so that quantities can be grouped. For example, the quantities length, time, and mass vary in nature, so that they have various dimensions. It is important to distinguish *dimension* from *unit*. A *physical quantity* has a particular *nature* (i.e., particular

Q.-M. Tan, *Dimensional Analysis*, DOI: 10.1007/978-3-642-19234-0_1,

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dimension) but a *unit* is a measure used to compare quantities. If two quantities have different natures (e.g., quantity of length and quantity of mass), dimensions of these quantities differ, cannot be compared profitably, and do not relate to unit. Conversely, if two quantities have similar dimensions (e.g., two lengths or two masses), *magnitudes* of these quantities can be compared profitably.

Comparing magnitudes of two quantities X_1 and X_2 with similar dimensions allows three possibilities:

$$1.\ \frac{X_1}{X_2} > 1,\ 2.\ \frac{X_1}{X_2} = 1,\ \text{and } 3.\ \frac{X_1}{X_2} < 1.$$

Denominator quantity X_2 is a viable standard unit for comparing X_1 and unit X_2 may be denoted as U:

$$X_1 = \frac{X_1}{X_2} \cdot X_2 = \frac{X_1}{X_2} \cdot U,$$

where ratio $\frac{X_1}{X_2}$ = exact magnitude of X_1 and $\frac{X_1}{X_2}$ = *dimensionless* pure number.

The above three possibilities may be written so that comparison unit = quantity X_1:

$$1.\frac{X_2}{X_1} < 1 \quad 2.\frac{X_2}{X_1} = 1, \text{and } 3.\frac{X_2}{X_1} > 1.$$

Clearly, unit is a quantity used for comparison and is not the nature of a quantity or the dimension of a quantity.

1.3 Quantities: Dimensional, Dimensionless, Fundamental, and Derived

Physical quantities that relate to a phenomenon or problem can be *dimensional quantities* or *dimensionless quantities*. For dimensional quantities, *magnitude* depends on the *unit* selected (e.g., unit of length, unit of time, unit of mass, and unit of force). For dimensionless quantities, *magnitude* usually depends on ratio of two quantities with same dimensions (e.g., ratio of different lengths, ratio of different times, and ratio of different forces or ratio of different energies). Physical quantities related to a phenomenon or problem can be divided into two systems: *fundamental quantities* and *derived quantities*. A system of fundamental quantities is such that the dimensions of the quantities in this system are mutually independent. Another system of derived quantities is such that the dimension of every quantity in this system can be expressed by a combination of the dimensions of fundamental quantities in the problem.

1.4 Measurement of Physical Quantities

A physical law or principle is determined through experiment or theoretical work. Through analysis and synthesis, quantities related to physical problems are examined, classified, and related. Correctness is verified according to whether or not the deduction conforms to experiments and observations.

Describing the physical phenomena requires measuring each quantity involved. A certain quantity of the same nature is taken as an appropriate standard unit of measurement so that quantities can be measured and compared. For example, if length L is to be measured, length L_0 can be taken as a unit in order to compare L, producing the magnitude of L, which can be denoted as l, i.e.,

$$L/L_0 = l, \quad \text{or } L = lL_0. \tag{1.1}$$

Measurement lacks absolute precision because of error related to technique, apparatus or the carefulness of the operator. For example, a gauge made of iridoplatinum was used for a long time to define the length of one meter. Later, precision improved and the length of one meter was redefined as 1,650,763.73 times the wavelength of the orange line of the isotope krypton.

In general, *error* Δl can be estimated and it is usual to represent Δl as a positive number. If physical quantity A is measured using quantity U as a measurement unit, magnitude can be a, but exact value is a_0 and error is Δa (>0):

$$A = a\,U \tag{1.2}$$

and

$$a = a_0 \pm \Delta a,\ a/a_0 = 1 \pm \Delta a/a_0$$

Performing addition, subtraction, multiplication, and division on magnitudes a and b for quantities A and B, where $b = b_0 \pm \Delta b$ and $b/b_0 = 1 \pm \Delta b/b_0$, produces dimensional relationships and dimensionless relationships:

1. $$a + b = (a_0 + b_0) \pm (\Delta a + \Delta b),$$
$$(a + b)/(a_0 + b_0) = 1 \pm (\Delta a + \Delta b)/(a_0 + b_0),$$

2. $$a - b = (a_0 - b_0) \pm (\Delta a + \Delta b),$$
$$(a - b)/(a_0 - b_0) = 1 \pm (\Delta a + \Delta b)/(a_0 - b_0),$$

3. $$a \cdot b = (a_0 \cdot b_0) \pm (a_0\Delta b + b_0\Delta a),$$
$$(a \cdot b)/(a_0 \cdot b_0) = 1 \pm (\Delta a/a_0 + \Delta b/b_0),$$

4. $$a/b = (a_0/b_0) \pm (a_0\Delta b + b_0\Delta a)/b_0^2,$$
$$(a/b)/(a_0/b_0) = 1 \pm (\Delta a/a_0 + \Delta b/b_0).$$

1.5 The Simple Pendulum

To show how *dimensional analysis* can reveal the essentials and inherent causality of a problem, it is useful to consider an idealized pendulum consisting of a weightless string of fixed length l and a small sphere of mass m (Fig. 1.1). In this case, the sphere is attached to the lower end of the string and the upper end of the string is fixed to a ceiling. Due to *gravity*, the sphere with initial deviation angle α oscillates around the plumb line at the fixed point on the ceiling within a definite *period* T_p.

Several approximation assumptions apply in the case of a *simple pendulum*:

1. Mass of the string is assumed << mass of the small sphere m.
2. Deformation of the string is assumed length l.
3. Compared to gravity, aerodynamic drag is assumed to be negligible.

Clearly, oscillation period T_p depends on four *governing parameters*:

1. mass of small sphere m;
2. length of string l;
3. gravitational acceleration g;
4. initial deviation angle α, thus:

$$T_p = f(m, l, g, \alpha). \tag{1.3}$$

The pendulum is a simple mechanical system. *Independent variables* of function f include three *fundamental quantities* having dimensions: m has dimension mass, l has dimension length, and g has dimension acceleration. A fourth independent variable α is dimensionless because an angle is defined by the ratio of two lengths. The dimension of *dependent variable* T_p is time, which can be expressed by combining the dimension of fundamental quantities l and g (length/acceleration)$^{1/2}$. Thus, T_p is *derived quantity*.

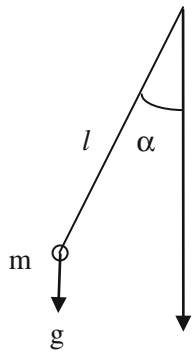

Fig. 1.1 A simple pendulum

Fundamental quantities m, l, and g can serve as a unit system for measuring quantities related to the pendulum problem, so according to the relationship (1.3),

$$T_p \Big/ (l/g)^{1/2} = f(1, 1, 1, \alpha).$$

$T_p \Big/ (l/g)^{1/2}$ varies only with α, so $T_p \Big/ (l/g)^{1/2}$ is a function of α:

$$T_p \Big/ (l/g)^{1/2} = f_1(\alpha).$$

Discarding subscript "1" for function f_1 produces:

$$T_p \Big/ (l/g)^{1/2} = f(\alpha) \tag{1.4}$$

In expression (1.4), $f(\alpha)$ means that $T_p \Big/ (l/g)^{1/2}$ is a function of α, not that $f(\alpha)$ is identical with $f(m, l, g, \alpha)$. Henceforth, f written without subscript indicates a functional relationship between dependent and independent variables, without specifying the particular function for such relationship.

Several conclusions appear:

1. T_p is proportional to $l^{1/2}$.
2. T_p is inversely proportional to $g^{1/2}$.
3. T_p does not depend on m.
4. $T_p \Big/ (l/g)^{1/2}$ depends only on α. (The particular form of $f(\alpha)$ should be determined by experiment or theoretical analysis.)

Using direct experiments to seek the particular form of $T_p = f(m, l, g, \alpha)$ means that if each of the four independent variables requires ten experiments, then a total of 10^4 experiments are required. Using dimensional analysis, ten experiments can determine $f(\alpha)$. Furthermore, when initial deviation angle α is sufficiently small, i.e., $\alpha << 1$, the problem becomes much simpler. Judging from the physical point of view, $f(\alpha)$ must be an even function that can be expressed as a Taylor's series at $\alpha = 0$:

$$f(\alpha) = f(0) + f''(0) \cdot \alpha^2 + f^{(4)}(0) \cdot \alpha^4 / (4!) + \cdots \cong f(0)$$

There can also be approximate formulation of satisfactory precision:

$$T_p = (l/g)^{1/2} \cdot f(0) \tag{1.5}$$

A single experiment can determine the value of constant $f(0)$, but this constant is derived from theoretical analysis: $f(0) = 2\pi$.

1.6 Essential Principles

Constructing a model and deriving results for the example of the pendulum involves *essential principles* of dimensional analysis:

Principle 1. Only magnitudes of quantities of similar dimension can be compared.

In constructing a model, several important assumptions underlie this principle. For example, the mass of the string is assumed to be insignificant compared with the mass of the small sphere and string deformation is assumed to be insignificant compared with string length.

Principle 2. Physical phenomena and physical laws do not depend on the unit system selected.

Identifying a geometric figure provides a simple example of Principle 2. A triangle consists of sides l_1, l_2, and l_3 and regardless of the distance separating the triangle and an observer, shapes observed are similar because these shapes belong to the same class and any side of a triangle can be selected as a unit of length. For example, taking l_1 as a unit to measure, l_2 and l_3 provides magnitudes l_2/l_1 and l_3/l_1, respectively. Two such dimensionless quantities can be used to clarify these triangles, regardless of how far the observer is from what is observed. This method for identifying geometric figures can also be used to identify physical phenomena and to understand physical laws involved in these phenomena.

While the dimension of length is confined to geometric figures, dimensions related to physical quantities can not only include length but also time, mass and others, so a selection of fundamental dimensional quantities can serve as units for measuring variables related to physical phenomena. Properly selecting dimensionless quantities including dependent and independent variables, can be even more meaningful and essential than dimensional variables, allowing causality of a problem to be reformulated as a dimensionless expression. In multiple cases in the same class, not only in a few special cases, dimensionless causal relationship is more concise and more objective than dimensional relationship in reflecting the essentials of the physical phenomena.